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Full Causal Theory of Bulk Viscosity and Specific Entropy of the Universe

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This article deals with the full Israel–Stewart causal theory of bulk viscosity as employed to the dissipative expansion of the early universe. It is shown that the nontruncated full theory can be cast in the form of a noncausal theory with an auxiliary condition which states that the square of dissipative contribution to the speed of sound varies with the particle number in a comoving volume. Also, a generalized temperature appears in a cosmological invariant which is shown to hold good for the dissipative expansion in an intermediate brief transition period (around the epoch time $\alpha = 10^{-23}$ s) between the very early "mild inflation" stage of the universe and the standard radiation-dominated FRW era of it. With this generalized temperature, the Gibbs equation has been generalized. This equation is also shown to have an alternative form with a term depending on bulk viscosity. In the dissipative transition period, the universe as a thermodynamically open system of viscous fluid can generate specific entropy. In this period the temperature rose to a considerable extent. Due to the cosmological invariant, the dissipative contribution to the speed of sound and consequently the particle number decreased sharply, ensuring the second law of thermodynamics. It is possible to have an estimate of the specific entropy in consistency with the observations. The total entropy and the particle number of the observable universe have also been found here. These estimates agree with the accepted values for them.

1. INTRODUCTION

In the early universe dissipative processes played an important role. One of such processes may be included in a theory of the evolution of the universe if one has to account for the present large value of the specific entropy per baryon. These processes are phenomenologically realized in a cosmic imperfect fluid model with bulk viscosity. In order to have a model causal and stable, the best option for a consistent relativistic nonequilibrium thermodynamics that may be employed is the full causal thermodynamics (Hiscock and Lindblom, 1983; Hiscock and Salmonson, 1991; Zakari and Jou, 1993) rather than the truncated version of it or the old first-order relativistic theory of Eckart (1940).

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Earlier we (De, 1997a, 1999) have considered the generation of specific entropy per baryon in the framework of the noncausal theory of Eckart in a cosmic viscous fluid model. Several other authors have considered FRW cosmologies in a noncausal and unstable first-order thermodynamics as well as in the "extended" irreversible thermodynamics, the truncated version of the full causal theory (for a good number of references see Maartens, 1995). In fact, the extended irreversible thermodynamics is a theory that includes the previously neglected second-order terms in Eckart theory. This can prevent noncausal and unstable behavior of the first-order theory. But, the truncated version of the causal thermodynamics of bulk viscosity does not behave properly in the late universe as in the case of noncausal theory. On the other hand, the nontruncated, that is the full causal theory does behave well at all cosmological times. Therefore it is desirable to employ full causal evolution equation for the bulk viscosity.

Presently we shall discuss the production of specific entropy in the early universe to account for the present observed value of it in the framework of full causal thermodynamics. Here, the early universe in its initial stage is regarded as a thermodynamically open system in the sense of Prigogine (1989), in which both the particle creation and the entropy generation are possible. In fact, it was shown earlier (De, 1999) that the universe went through a "mild" inflation from a Planck-order time to an epoch $\alpha = 0.26 \times 10^{-23}$ s with particle and entropy production. It had a very brief "transition" period just before the epoch α, when the universe became an imperfect fluid with bulk viscosity. In this transition period the particles (massive) became relativistic and contributed to the radiation energy density. This phase transition in the transition period is, in fact, modelled in terms of the dissipative process which is realized in the viscous fluid model of the universe. After the period of transition the universe entered into the FRW radiation-dominated era with the standard cosmology.

We begin with the following section in describing the cosmic fluid of the transition period with the full causal evolution equation for the bulk viscosity. In Section 3, we shall discuss in more details both the truncated and the full transport equations and show that the expression of the bulk viscous pressure for the full theory retains the same form as that in the linear theory if the square of dissipative contribution to the speed of sound is proportional to the particle number in a comoving volume. A cosmological invariant is shown to hold good with a generalized temperature and with this temperature the Gibbs equation is generalized. In an alternative generalization with the usual temperature it will be shown there that this equation has an additional term depending on bulk viscosity. An expression for the rate of production of specific entropy will also be given. In the subsequent Section 4, the production of specific entropy in the transition period will be discussed and an estimate of it will be given. We shall also find the estimates of total entropy and particle number of the observable universe. In the final Section 5, a summary of main conclusions and results of the article will be furnished with some remarks. In the following we shall use the natural unit with Boltzmann's constant $k_B = 1$.

2. COSMIC FLUID WITH BULK VISCOSITY

The energy momentum tensor for an imperfect cosmic fluid without shear viscosity or heat flow is given by

$$T_{ab} = \rho u_a u_b + (p + \Pi)h_{ab} \tag{1}$$

where u^a is the particle-frame 4-velocity, ρ the energy density, and p the equilibrium pressure. Here, Π is the bulk viscous stress and $h_{ab} = g_{ab} + u_a u_b$ represents the projection tensor. Since in an expanding universe with the viscous fluid the dissipation due to this viscous stress gives rise to a decrease in kinetic energy and a consequent decrease in pressure, then we must have $\Pi \leq 0$. It is discussed in Udey and Israel (1982) that the dissipation from microscopic heat flow due to different cooling rates in the mixture (for example, radiation and matter, low- and high-energy particles in Boltzmann gas etc.) can be represented by the viscous stress Π. The particle flow vector and the entropy 4-flux are respectively given by the following relations.

$$N^a = nu^a \tag{2}$$

where n is the particle number density.

$$S^a = \sigma_{\text{eff}}\, N^a \tag{3}$$

where σ_{eff} is the effective nonequilibrium specific entropy which must be positive. σ_{eff} is given by (Hiscock and Salmonson, 1991; Zakari and Jou, 1993)

$$\sigma_{\text{eff}} = \sigma - \frac{\tau}{2nT\zeta}\Pi^2 \tag{4}$$

where T and σ are the local equilibrium temperature and specific entropy respectively. Here, ζ and τ are respectively the coefficient of bulk viscosity and the relaxation time. From (3) and (4) it follows that

$$\begin{aligned} S^a{}_{;a} &= \left[\left(\sigma - \frac{\tau}{2nT\zeta}\Pi^2\right) N^a\right]_{;a} \\ &= \dot{\sigma} n + \sigma(\dot{n} + 3Hn) - \frac{\Pi}{T}\left[\frac{\tau}{\zeta}\dot{\Pi} + \frac{1}{2}\Pi T\left(\frac{\tau u^a}{\zeta T}\right)_{;a}\right] \end{aligned} \tag{5}$$

where $3H = u^a{}_{;a}$ is the fluid expansion and $\dot{n} = n_{;a}\, u^a$. H defines a comoving length scale R by the relation $\dot{R}/R = H$. This length scale is the cosmic scale factor in a FRW universe. Now, if the conservation laws $N^a{}_{;a} = 0$ and $T^{ab}{}_{;b} = 0$

which imply respectively,

$$\dot{n} + 3Hn = 0 \quad \text{and} \quad \dot{\rho} + 3H(\rho + p + \Pi) = 0 \tag{6}$$

hold, then we have from (5) the following equation for the rate of entropy production:

$$S^a{}_{;a} = \dot{\sigma} n - \frac{\Pi}{T}\left[\frac{\tau}{\zeta}\dot{\Pi} + \frac{1}{2}\Pi T\left(\frac{\tau u^a}{\zeta T}\right)_{;a}\right] \tag{7}$$

Again, the Gibbs equation for the local equilibrium variables σ and T is

$$T\,d\sigma = d\left(\frac{\rho}{n}\right) + p\,d\left(\frac{1}{n}\right) \tag{8}$$

From Eq. (8) and the conservation equations (6), the following equation for the production rate of specific entropy follows:

$$\dot{\sigma} = -\frac{3H\Pi}{nT} \tag{9}$$

Using (9) we get from (7) the entropy production rate as

$$S^a{}_{;a} = -\frac{\Pi}{T}\left[3H + \frac{\tau}{\zeta}\dot{\Pi} + \frac{1}{2}\Pi T\left(\frac{\tau u^a}{\zeta T}\right)_{;a}\right] \tag{10}$$

It is shown in Hiscock and Lindblom (1983) (see also Hiscock and Salmonson, 1991; Zakari and Jou, 1993) that the following causal evolution equation for bulk viscosity (the transport equation) can ensure the second law of thermodynamics $S^a{}_{;a} \geq 0$ in a simplest possible way, that is, in a linear form of equation for the bulk viscous pressure Π:

$$\Pi + \tau\dot{\Pi} = -3\zeta H - \frac{\epsilon}{2}\tau\Pi\left(3H + \frac{\dot{\tau}}{\tau} - \frac{\dot{\zeta}}{\zeta} - \frac{\dot{T}}{T}\right) \tag{11}$$

Here, $\epsilon = 1$ gives the full theory. On the other hand, $\epsilon = 0$ makes it a truncated theory. It is pointed out here that $\tau = 0$ gives the noncausal theory. The transport equation (11) leads to

$$S^a{}_{;a} = \frac{\Pi^2}{\zeta T} \tag{12}$$

It is pointed out by Maartens (1995) that the truncated theory implies an implicit "temperature law"

$$T \propto \frac{\tau}{\zeta}R^3 \tag{13}$$

Of course, Maartens has argued that this temperature law can make the situation unphysical because the temperature must be increasing as $R \to \infty$, that is, in the late universe unless the ratio τ/ζ can decrease faster than the increase of the comoving volume. For this reason he advocated for using either full nontruncated form of the transport equation or a generalized nonequilibrium temperature and pressure with the generalized Gibbs equation (see Gariel and Le Denmat, 1994; Pavon *et al.*, 1982; Zakari and Jou, 1993).

3. GENERALIZED GIBBS EQUATION

We first consider one important aspect that appears not to have been pointed out previously. The fact is that one can retain a solution of the full transport equation (11) (with $\epsilon = 1$) in the same form as that in the noncausal theory, that is,

$$\Pi = -3H\zeta \tag{14}$$

if the following relation is satisfied as the supplement:

$$3H + \frac{\dot{\tau}}{\tau} - \frac{\dot{\zeta}}{\zeta} - \frac{\dot{T}}{T} + \frac{2\dot{\Pi}}{\Pi} = 0 \tag{15}$$

or, equivalently,

$$\frac{V\zeta^2}{v^2 T} = \text{constant} \tag{16}$$

where $V = R^3(t)$ is the comoving volume. In deducing (16) from (15) we have taken into account the following field equation of the flat FRW universe:

$$k\rho = 3H^2, \quad (k = 8\pi G) \tag{17}$$

and also the relation (Maartens, 1996) between the relaxation time τ and the coefficient of bulk viscosity ζ given by

$$\tau = \frac{\zeta}{v^2\gamma\rho} \tag{18}$$

where v is the dissipative contribution to the speed of sound and satisfies the inequality.

$$v^2 \leq 2 - \gamma \tag{19}$$

because of causality. In deducing (18) Maartens has used the γ-law equation of state

$$p = (\gamma - 1)\rho \tag{20}$$

We have previously considered the early universe as a thermodynamically open system in which the creation of particles could occur (De, 1993a, 1999). The law of conservation of particles, that is, $N^a{}_{;a} = 0$ does not hold for such systems.

On the other hand, the other conservation law in (6) has been modified there in the following form (for the bulk viscous fluid model):

$$\dot{\rho} - \frac{\dot{n}}{n}(\rho + p) + 3H\Pi = 0 \tag{21}$$

It should be noted that when the particle number is conserved we have $\dot{n}/n = -3H$ and the usual conservation law in (6) follows. Now, the entropy equation for the viscous fluid is

$$T\frac{\dot{S}}{V} = \frac{TS}{V}\frac{\dot{N}}{N} - 3H\Pi \tag{22}$$

This is a generalization of the entropy equation for the open thermodynamic system of perfect fluid, given by Prigogine (1989). Here, S is the entropy in a comoving volume, that is, $S = sR^3$ where s is the entropy density. Also, $N = nR^3$. Now, since the specific entropy is given by

$$\sigma = \frac{S}{N} = \frac{s}{n} \tag{23}$$

the same Eq. (9) for the production rate of it follows from the entropy equation (22). Again, using (9) and the conservation equation (21) for the present case of the open system we can find Gibbs equation (8) as follows:

$$d\sigma = -\frac{3H\Pi}{nT}\,dt = \frac{dt}{nT}\left\{\dot{\rho} - \frac{\dot{n}}{n}\,(\rho + p)\right\}$$

or

$$T\,d\sigma = \frac{d\rho}{n} - \frac{dn}{n^2}(\rho + p) = d\left(\frac{\rho}{n}\right) + p\,d\left(\frac{1}{n}\right)$$

For the present case where the particle number is not conserved, the rate of entropy production, that is, $S^a{}_{;a}$ is given by (5) in which $\dot{\sigma}$ is to be replaced by its expression as in (9). Usually the cosmic fluid is locally described by an equilibrium equaiton of state that has the approximated form of an ideal gas:

$$p = nT \tag{24}$$

with the linear barotropic equation of state (20).

Also, for a flat FRW universe with "power-law" expansion (that is, $R(t) \propto t^\beta$), it is implied from (14), (20), and (21) that $\zeta(t) \propto 1/t \propto \rho^{1/2}$
or

$$\zeta(t) = A\rho^{1/2} \quad \text{(say)} \tag{25}$$

if, of course, the number density n also follows the power-law. When the particle number is conserved, that is, when $N^a{}_{;a} = 0$ we have $\dot{n}/n = -3H$ and n follows the power-law. Conversely, when $\zeta(t)$ is of the form (25) then n follows the power-law also for the case of nonconserved particle number.

Now, from (16) and (24) it follows that

$$\frac{N\zeta^2}{v^2 p} = \text{constant}$$

or

$$\frac{A^2 N}{v^2(\gamma - 1)} = \text{constant} \quad (\gamma \neq 1)$$

or

$$N \propto v^2 \tag{26}$$

Thus, the full causal theory gives rise to noncausal form as the solution for the viscous stress with the implicit condition (26). From this condition it follows that the particle number conservation implies a “constant” (v) dissipative contribution to the speed of sound. Further, for radiation-dominated FRW universe we have from (16) and (25),

$$R^3\rho \propto v^2 T \tag{27}$$

Since, in this case $\rho \propto 1/R^4$ it follows that

$$v^2 T R = \text{constant} \tag{28}$$

This relation insists us to generalize the temperature as

$$\tilde{T} = \ell v^2 T \tag{29}$$

so that one can have the usual cosmological invariant

$$\tilde{T} R = \text{constant} \tag{30}$$

Here, ℓ is a constant which will be specified later. In this connection we can remind the following result given in Maartens (1996). If one insists $p = nT$ then the Stefan–Boltzmann law $\rho \propto T^4$ cannot hold. Alternatively, if we impose the Stefan–Boltzmann law then the ideal gas law cannot be valid unless the fluid returns to equilibrium. For the present case, $p = nT$ and $\rho \propto T^4$ cannot hold simultaneously. In fact, $\rho \propto v^8 T^4$. Again, with the generalized temperature, it follows that

$$\rho \propto \tilde{T}^4 \tag{31}$$

but $p \neq n\tilde{T}$. On the other hand,

$$p = \frac{n\tilde{T}}{\ell v^2} \tag{32}$$

These two laws can hold simultaneously when the fluid returns to equilibrium with the particle number conservation, that is, with a constant v.

With the generalized temperature, the Gibbs equation and the entropy equation (22) are, respectively, generalized into the following forms (by replacing T by $\tilde{T}$):

$$\tilde{T}\, d\sigma = d\left(\frac{\rho}{n}\right) + p\, d\left(\frac{1}{n}\right) \tag{33}$$

and

$$\frac{\tilde{T}\dot{S}}{V} = \left(\frac{\tilde{T}S}{V}\right)\frac{\dot{N}}{N} - 3H\Pi \tag{34}$$

From (33), it follows, by using the conservation law (21), that

$$\dot{\sigma} = \frac{-3H\Pi}{n\tilde{T}} = \frac{-3H\Pi}{nT\ell v^2} \tag{35}$$

as the rate of increase in specific entropy. This relation also follows directly from the generalized entropy equation (34). With Eq. (35), we have the entropy production rate from (5) as

$$S^a{}_{;a} = -\frac{3H\Pi}{\tilde{T}} + \sigma(\dot{n} + 3Hn) - \frac{\Pi}{T}\left[\frac{\tau}{\zeta}\dot{\Pi} + \frac{1}{2}\Pi T\left(\frac{\tau u^a}{\zeta T}\right)_{;a}\right] \tag{36}$$

or, by using (26) which give

$$\frac{\dot{N}}{N} = \frac{\dot{n}}{n} + 3H = \frac{2\dot{v}}{v} \tag{37}$$

we have

$$S^a{}_{;a} = -\frac{3H\Pi}{\tilde{T}} + 2\sigma n\frac{\dot{v}}{v} - \frac{\Pi}{T}\left[\frac{\tau}{\zeta}\dot{\Pi} + \frac{1}{2}\Pi T\left(\frac{\tau u^a}{\zeta T}\right)_{;a}\right] \tag{38}$$

That this generalization of Gibbs equation does not change the tenet of the full transport equation given by (11) will be realized from the following consideration which shows, that the second law of thermodynamics, $S^a{}_{;a} \geq 0$, can also be ensured in this case. From (36), using the full transport equation (11) we have

$$\begin{aligned} S^a{}_{;a} &= -\frac{3H\Pi}{\tilde{T}} + \sigma n\left(3H + \frac{\dot{n}}{n}\right) + \frac{\Pi}{T}\left(\frac{\Pi}{\zeta} + 3H\right) \\ &= \frac{9H^2\zeta}{\tilde{T}} + \sigma n\left(3H + \frac{\dot{n}}{n}\right) \quad \text{[using (14)]} \\ &= \frac{\zeta\theta^2}{\tilde{T}} + \sigma n\left(\theta + \frac{\dot{n}}{n}\right) \end{aligned} \tag{39}$$

where $3H = \theta$ is the expansion scalar. Now, we use the following relation for the specific entropy which is the generalization (by replacing T by $\tilde{T}$) of it (Israel and

Stewart, 1979; Prigogine, 1989) in terms of relativistic chemical potential μ:

$$\sigma = \frac{h - \mu n}{T \ell v^2 n} = \frac{h - \mu n}{\tilde{T} n} \tag{40}$$

where $h = \rho + p$ is the enthalpy per unit volume. Then, from (39) we have

$$S^a{}_{;a} = \frac{\zeta \theta^2}{\tilde{T}} + \frac{h - \mu n}{\tilde{T}} \left(\theta + \frac{\dot{n}}{n} \right) \tag{41}$$

From (14) and (21) we have

$$\dot{\rho} - \frac{\dot{n}}{n} h - \zeta \theta^2 = 0 \tag{42}$$

Thus, we have from (41) and (42) the following expression for the rate of entropy production:

$$S^a{}_{;a} = \frac{1}{\tilde{T}} \left\{ \dot{\rho} + h\theta - \mu n \frac{\dot{N}}{N} \right\} \quad \text{[by using (37)]} \tag{43}$$

Again with the equation of state (20), we have

$$S^a{}_{;a} = \frac{\rho}{\tilde{T}} \left\{ \frac{\dot{\rho}}{\rho} \theta - \frac{\mu n}{\rho} \frac{\dot{N}}{N} + \gamma \theta \right\} \tag{44}$$

For the FRW universe with $R(t) \propto t^{1/2}$ and $\gamma = 4/3$, we have

$$\frac{\dot{\rho}}{\rho} + \gamma \theta = \frac{2\dot{\theta}}{\theta} + \frac{4}{3} \theta = 0$$

Consequently, we get

$$S^a{}_{;a} = -\frac{\mu n}{\tilde{T}} \frac{\dot{N}}{N} \tag{45}$$

Thus, the second law of thermodynamics, $S^a{}_{;a} \geq 0$, can be ensured if $\dot{N}/N \leq 0$. For $\mu = 0$, the equality sign in (45), of course, holds good. For $\mu \neq 0$, the particle number cannot increase for the validity of the second law of thermodynamics. This situation will be examined in the subsequent section.

Interestingly, the generalized Gibbs equation (33) can be written in an alternative form:

$$T \, d\sigma = d \left(\frac{\rho}{\hat{n}} \right) + p \, d \left(\frac{1}{\hat{n}} \right) + \frac{2h}{\hat{n}} \frac{dv}{v} \tag{46}$$

Here, T is the usual local equilibrium temperature but the particle number density is generalized as

$$\hat{n} = \ell v^2 n \tag{47}$$

It is evident that the additional term in the Gibbs equation (46) depends on the bulk viscous pressure through the parameter v. It may be pointed out here that the generalized Gibbs equation in Pavon *et al*. (1982), Gariel and Le Denmat (1994), and Zakari and Jou (1993) has an additional term that also depends on the bulk viscosity. It is interesting to note that the generalized entropy equation (34) takes the following form:

$$\frac{T\dot{\hat{S}}}{V} = \frac{T\hat{S}}{V}\frac{\dot{\hat{N}}}{\hat{N}} - \theta\Pi \tag{48}$$

with the generalized particle number $\hat{N} = \hat{n}R^3$ and the generalized entropy $\hat{S} = S\ell v^2$ in the comoving volume R^3. Obviously, the form of the entropy equation remains invariant under the transformation to the generalized quantities (particle number and entropy). Again, the specific entropy σ is given by

$$\sigma = \frac{s}{n} = \frac{\hat{s}}{\hat{n}} = \frac{\hat{S}}{\hat{N}} = \frac{S}{N} \tag{49}$$

and its rate of increase can be written as

$$\dot{\sigma} = -\frac{3H\Pi}{n\tilde{T}} = -\frac{3H\Pi}{n\ell v^2 T} = -\frac{3H\Pi}{\hat{n}T} \tag{50}$$

When the particle number becomes constant, that is, $N^a{}_{;a} = 0$ we have $v =$ constant; the Gibbs equation as well as the entropy equation take their usual forms.

Now, from (14) and the conservation law (21) it follows that

$$\zeta(t) = \frac{1}{3k}\left(\frac{2\dot{\theta}}{\theta} + \gamma\theta - \frac{2\gamma\dot{v}}{v}\right) \tag{51}$$

In deriving this, we have used field equation (17), equation of state (20), and Eq. (37). Here also, for the FRW universe, with $R(t) \propto t^{1/2}$ and $\gamma = 4/3$ we have, since

$$\frac{2\dot{\theta}}{\theta} + \gamma\theta = 0, \qquad \zeta(t) = -\frac{8}{9k}\frac{\dot{v}}{v} \tag{52}$$

and

$$\dot{\sigma} = -\frac{8\theta^2}{9knT\ell}\frac{\dot{v}}{v^3} \tag{53}$$

Again, by using (20) and (24) we have

$$\dot{\sigma} = -\frac{8\dot{v}}{\ell v^3} \tag{54}$$

For the case $\gamma \neq 4/3$, we have, of course, the following rate of increase for the specific entropy:

$$\dot{\sigma} = \frac{1}{(\gamma - 1)v^2\ell} \left\{ \frac{2\dot{\theta}}{\theta} + \gamma\theta - \frac{2\gamma\dot{v}}{v} \right\} \quad (\gamma \neq 1) \tag{55}$$

For a more general case of time dependent γ we have to modify some of the equations above. Of them, the important one is (26) which in this case becomes

$$N \propto \gamma(\gamma - 1)v^2 \quad (\gamma \neq 1) \tag{56}$$

The rate of increase in specific entropy for this case is given by

$$\dot{\sigma} = \frac{1}{(\gamma - 1)v^2\ell} \left\{ \frac{2\dot{\theta}}{\theta} + \gamma\theta - \dot{\gamma}\,\frac{2\gamma - 1}{\gamma - 1} - 2\gamma\frac{\dot{v}}{v} \right\} \quad (\gamma \neq 1) \tag{57}$$

4. SPECIFIC ENTROPY: MILD INFLATION AND TRANSITION ERA

In the previous articles (De, 1993a, 1999), both the particle and the entropy productions were considered in an early universe taken as a thermodynamically open system. Such an early universe as a thermodynamically open system was considered earlier by Prigogine (1989), who modified the thermodynamic energy conservation law for homogeneous and isotropic universes. An early universe with bulk viscosity was also considered previously (De, 1997b, 1999) in the framework of standard general relativity (GR) as well as in a modified general relativity (MGR) of Rastall (1972), Al-Rawaf and Taha (1996a,b). In the imperfect fluid model of the early universe with bulk viscous pressure it was possible there to obtain the present value of the specific entropy per baryon in consistency with the observation. Of course, the relativistic theory of non-equilibrium thermodynamics employed therein for the FRW universe filled with dissipative fluid was the noncausal theory. Here, we shall reconsider the early universe as dissipative fluid described by the full theory of causal transport equation which is given and discussed in the preceding section.

It was shown in a previous article (De, 1999) that the universe went through a “mild inflationary era” from a Planck order time to an epoch $t \approx 10^{-23}$ s with no “turn-on” and “turn-off” problems for this stage. In that period, there were particle creation as well as entropy production although any increase in the specific entropy was not possible. This fact will also be evident later in the following discussion. There the specific entropy was shown to be produced in a “transition” period just before the epoch time $\alpha = 0.26 \times 10^{-23}$ s, when the fluid became dissipative due to transition of the particles (massive) into the ultra relativistic state around this epoch α. In that consideration the epoch-dependence of particle masses, significant only in the early period of evolution of the universe was taken into account. The particle masses which were very large (of the order of Planck mass) attained their "near" present values at this transition era. The epoch-dependence of particle masses is, in fact, a consequence of Finsler geometric approach in building up the internal symmetry of hadrons (De, 1986, 1991, 1997a). This geometrical approach provides the field equations for the fundamental particles and also the "dynamics" of hadrons.

In the previous section it was pointed out that Prigogine (1989) modified the thermodynamic conservation law.

$$d(\rho R^3) + p\,d(R^3) - \frac{h}{n} d(nR^3) = dQ \tag{58}$$

For adiabatic transformation we have $dQ = 0$ and consequently we can find the following equation:

$$\dot{\rho} = \frac{\dot{n}}{n} h \tag{59}$$

This equation is, indeed, modified into (21) for imperfect fluid with bulk viscosity. The field equation (17) holds for the mild inflationary stage as well as for transition era. In an alternative interpretation, one can retain the usual conservation law (Bianchi identity) with a phenomenological pressure $\hat{p}$ instead of above true thermodynamic pressure p, that is,

$$d(\rho R^3) = -\hat{p}\,d(R^3) \tag{60}$$

where the two pressures $\hat{p}$ and p are related by

$$\hat{p} = p + p_c \tag{61}$$

Here, p_c represents a pressure that corresponds to the creation of particle if it is negative. When $p_c = 0$ the creation of particles stops and in this case $\hat{p} = p$. Consequently, the conventional law of conservation holds, or in otherwords, the usual Einstein equations of GR hold. The pressure p_c is given by

$$p_c = -\frac{h}{\theta}\frac{\dot{N}}{N} \tag{62}$$

For this particle production era of the very early universe the energy density $\rho = \rho_m + \rho_\gamma$ should be dominated by the matter density ρ_m, that is, the radiation energy density $\rho_\gamma \ll \rho_m$. Consequently, we can take

$$\rho \simeq \rho_m = mn \tag{63}$$

where, m is the particle mass. Here, for simplicity, the masses of created particles of all types are assumed to be the same. The mass m is, on the other hand, epoch-dependent and the following relation gives rise to the mass of a particle at an epoch time t:

$$m = \bar{m}\left(1 + \frac{2\alpha\theta}{3}\right) \tag{64}$$

where $\alpha = 0.26 \times 10^{-23}$ s (De, 1997a) and $\bar{m}$ represents the "inherent" mass of the particle. This inherent mass is equal to the present mass of the particle with an extremely high degree of accuracy. For massless particles (that is, for particles with no inherent mass) the corresponding relation is

$$m = \left(\frac{2}{3}\right)\alpha\hat{m}\theta \tag{65}$$

when $\hat{m}$ is the mass of the particle at the epoch $t = \alpha$. The equation of state is taken to be barotropic and is of the form:

$$p = f(\theta)\rho \tag{66}$$

From Eqs. (59), (63), (64), (66), and (17) the following governing equations for the early universe can be obtained.

$$\frac{\dot{\rho}}{\rho} = \frac{2\dot{\theta}}{\theta} = \frac{\dot{n}}{n}(1 + f(\theta)) = \frac{\dot{m}}{m}\left(1 + \frac{1}{f(\theta)}\right) \tag{67}$$

$$\frac{\dot{m}}{m} = \frac{2\alpha\dot{\theta}}{3 + \alpha\theta} \tag{68}$$

For a viscous fluid model of the universe, more general governing equations for the evolution of the early era can be obtained with the use of the corresponding conservation law (21) and the following expressions of ρ_m and ρ_γ (De, 1993a):

$$\rho_m = mn_m \tag{69a}$$

$$\rho_\gamma = (\Gamma - 1)\, mn_m + E_\gamma n_\gamma \tag{69b}$$

$$\frac{\rho_\gamma}{\rho_m} = \Gamma^2 - 1 \tag{69c}$$

where

$$\Gamma = \left(1 - \frac{\langle v^2\rangle}{c^2}\right)^{-1/2} \tag{69d}$$

and $n = n_m + n_\gamma$ is the total number density, n_m and n_γ being particle and photon number densities respectively. Here, $\langle v^2\rangle$ is the mean square velocity of the particles. The said governing equations are

$$\frac{\dot{\rho}}{\rho} = \frac{2\dot{\theta}}{\theta} = 3\,k\zeta(t) + (1 + f(\theta))\frac{\dot{n}}{n}$$
$$= \frac{1 + f(\theta)}{f(\theta)} \frac{E_\gamma}{E_\gamma + \Gamma(\Gamma - 1)m} \left\{ \frac{\dot{m}}{m} + \frac{\dot{E}_\gamma}{E_\gamma} \left[\frac{\Gamma(\Gamma - 1)m}{E_\gamma} \right] \right.$$
$$\left. + \frac{\dot{\Gamma}}{\Gamma} \left[2 - \frac{\Gamma m}{E_\gamma} \right] \right\} - \frac{3k\zeta(t)}{f(\theta)} \tag{70}$$

where we have used the expression (14) for bulk viscous pressure. These equations are, in fact, modifications of the governing equations obtained earlier for perfect fluid (De, 1993a)

The trivial solution of Eqs. (67) and (68) is the usual exponential inflation

$$\dot{\theta} = \dot{\rho} = \dot{n} = \dot{m} = 0 \tag{71}$$

On the other hand, if $\dot{\theta} \neq 0$, these equations give a solution for $f(\theta)$. It is given by

$$f(\theta) = \frac{\alpha\theta}{3 + \alpha\theta} \tag{72}$$

Thus, the equation of state is specified by these equations. For this nontrivial case the expansion scalar θ cannot be determined. It was, indeed, determined in a previous consideration (De, 1999) of the early universe in the framework of MGR. The expansion scalar for GR can also be obtained from the MGR case as a special case with the parameter value $\eta = 1$. This parameter η is regarded as the characteristic of the non-Newtonian regime and the standard GR corresponds to this particular value of the parameter. There the equation for θ which decides the nature of the early universe is obtained as

$$f(\theta)\dot{\theta}\left(\frac{2}{\theta} - \frac{(\eta - 2) f'(\theta)}{1 + (\eta - 1) f(\theta)}\right) + f'(\theta)\dot{\theta} + \frac{\theta}{3}[1 + f(\theta)] = 0 \tag{73}$$

The solution for θ for all η in its range, $0 < \eta \leq 1$ is $\theta = 3/t$ and hence $R(t) \propto t$. This corresponds to a "mild inflation" in the early stage of the universe. Also, it was shown there that the mild inflation turned off automatically around the epoch time α, when the universe entered into the radiation-dominated FRW stage with standard cosmology. This "inflation" has no "turn-on" problem as it was shown in a previous article (De, 1993b) that the universe must have gone to an expansion stage from a Minkowski space–time filled with created very massive particles (masses about 50 times Planck mass). In fact, these massive particles were created quantum mechanically through an anisotropic perturbation of that flat space–time. These very massive particles made the Minkowski space–time unstable and expanding (Nardone, 1989). In the mild inflationary stage thus turned on, the specific entropy was not produced as we shall see now. In fact, it is evident from (50) that a nonzero bulk viscous pressure can only produce the specific entropy. Since $\zeta(t) \geq 0$, we have from (51),

$$\frac{2\dot{\theta}}{\theta} + \gamma\theta - \frac{2\gamma\dot{v}}{v} \geq 0 \tag{74}$$

Now, for the "creation era" FRW universe with $R(t) \propto t^{\beta}(\beta > 0)$ before the epoch time α, we have from (63) and (68),

$$\frac{\dot{\rho}}{\rho} = \frac{2\dot{\theta}}{\theta} = \frac{\dot{m}}{m} + \frac{\dot{n}}{n} \simeq \frac{\dot{\theta}}{\theta} + \frac{n}{n}$$

therefore,

$$\frac{2\dot{v}}{v} = \frac{\dot{N}}{N} = \frac{\dot{n}}{n} + \theta = \frac{\dot{\theta}}{\theta} + \theta$$

Then the inequality (74) gives the condition

$$(2-\gamma)\frac{\dot{\theta}}{\theta} \geq 0$$

Now, since $\dot{\theta}/\theta = -1/t < 0$, then we must have $2 - \gamma \leq 0$. Again, since γ cannot be >2, we must have $\gamma = 2$. This value of γ is, indeed, obtained from (72) for $t < \alpha$. Now, this value of γ makes $\zeta(t) = 0$ and consequently it is concluded that no specific entropy was produced in this particle creation era of mild inflation.

Now, for the "creation era" of the early universe before the epoch α, we have, by using the equation of state (66) with $f(\theta) \simeq 1$ and the ideal gas law (24),

$$p = \rho = mn = nT \tag{75}$$

since this era is matter-dominated and the energy density is given by (63). Thus, we have from (75) with the mass given in (64) for the inflation period ($t < \alpha$),

$$T = m \simeq \frac{2\alpha\bar{m}}{t} \tag{76}$$

where $\bar{m}$ is the inherent mass of the particle. The equation (76) gives the temperature law for this period. Taking muons as the representative particles we have $\bar{m} = 5.37 \times 10^{12}\ \text{cm}^{-1}$ (in the unit $\hbar = c = k_b = 1$ with quantities expressed in cm only). Therefore, around the epoch α the temperature might have been $10^{13}\ \text{cm}^{-1}$. From (76), it follows that the temperature was $5 \times 10^{32}\ \text{cm}^{-1}$ at the Planck time. Thus, universe temperature dropped from its very large value at t_{pl} to a much lower value ($\approx 10^{13}\ \text{cm}^{-1}$) around the epoch α. In fact, at an epoch $t = 0.1\ \alpha$ the temperature was $10^{14}\ \text{cm}^{-1}$. On the other hand, we have calculated the energy density at the Planck order time $\hat{t} = 0.05\ t_{\text{pl}}$ from the quantum matter creation due to an anisotropic perturbation of the Minkowski space–time (De, 1993b). The universe went on an expansion phase (mild inflation) at this Planck order time $\hat{t}$ and the energy density followed the rule $\rho \propto 1/t^2$ due to the Einstein field equation (17). This can give the value of the energy density at the epoch time α. Actually, at $\hat{t}$, the energy density was calculated as

$$\rho(\hat{t}) = 2.8 \times 10^4\, m_{\rm pl}^4 \tag{77}$$

where $m_{\rm pl}$ is the Planck mass given by $G = 1/8\pi m_{\rm pl}^2$. Its value is $1.22 \times 10^{32}\,{\rm cm}^{-1}$. Then it follows that

$$\rho(\alpha) = \left(\frac{\hat{t}}{\alpha}\right)^2 \rho(\hat{t}) = 6.14 \times 10^{90}\,{\rm cm}^{-4} \tag{78}$$

Now, if the radiation-dominated FRW universe started at the epoch α with the standard cosmological principle,
then

$$\rho(\alpha) \simeq \rho_\gamma(\alpha) = \frac{\pi^2}{15} T^4 \tag{79}$$

From the relation (79) with $\rho(\alpha)$ in (78) the temperature at the epoch α can be determined. It is given by

$$T_\alpha = 5.52 \times 10^{22}\,{\rm cm}^{-1} \tag{80}$$

Thus, we see that the temperature of the universe dropped considerably in an epoch $\alpha'(\alpha' \lesssim \alpha)$ around the epoch α, when the matter-dominated mild inflation ended. The usual radiation-dominated FRW universe was set in just at the epoch α when the temperature shot up to a very large value compared to that at α'. In this transition from the "creation era" to the usual radiation era the universe might have been an imperfect fluid with bulk viscosity. It is a transition period (α', α) when the particles became relativistic and, in fact, this phase transition is modelled in terms of dissipative process of bulk viscosity.

Now, for an imperfect fluid the relation (61) is modified into the following relation:

$$\hat{p} = p + p_c + \Pi \tag{81}$$

where p_c is given by (62). For the full causal theory the viscous pressure Π is given by (14) with the supplementary condition (26). Again, the phenomenological pressure $\hat{p}$ is given as (De, 1993a)

$$\hat{p} = \frac{1}{3}\frac{\langle v^2\rangle}{c^2}\rho \tag{82}$$

Using (69d), we have $\hat{p} = (\rho/3)(1 - 1/\Gamma^2)$ and consequently the relation (81) gives

$$p + p_c + \Pi = \frac{\rho}{3}\left(1 - \frac{1}{\Gamma^2}\right) \tag{83}$$

For the transition period (α', α) when the particles became relativistic, Γ achieved a very large value since $\langle v^2 \rangle \to c^2$. Therefore, we have from (83) the following relation with the use of the equation of state (66):

$$p_c + \Pi + \rho\left(f(\theta) - \frac{1}{3}\right) = 0 \tag{84}$$

Again, for the transition era (α', α) we have, from (72), $f(\theta) \simeq 1/3$ since for this period $\theta = 3/2t$ as $\hat{p} = (1/3)\rho$. Consequently, we have

$$p_c + \Pi \simeq 0 \tag{85}$$

Since $\Pi < 0$, we must have $p_c > 0$ and therefore it is evident from (62) that the particle number must decrease in this transition period in consistency with the validity condition (45) of second law of thermodynamics. Using (14), (26), and (62) we have from (85),

$$\frac{8\rho}{3\theta}\frac{\dot{v}}{v} + \zeta\theta \simeq 0 \quad \left(\text{since } f(\theta) \simeq \frac{1}{3}\right)$$

Therefore we have, by using the field equation (17),

$$\zeta(t) \simeq -\frac{8}{9k}\frac{\dot{v}}{v}$$

which is the same relation (52) obtained in an alternative way. Consequently, we have the same relation (54) for the rate of increase in specific entropy. In fact, the expression in (54) is an approximation of this rate given in (57) for a general case of time-dependent γ because the other terms in the R.H.S. of that expression are small compared to the last term there. We have pointed out earlier that before the transition period $\dot{\sigma} = 0$. Again, if the usual radiation-dominated FRW era started at the epoch α we must have $p_c = 0$ and therefore $\dot{N} = 0 = \dot{v}$ for $t \geq \alpha$. Also, from (85) it follows that $\Pi = 0$ after this epoch. During the transition period v decreases since N decreases. If v^2 decreases more rapidly than the increase in $R(t)$ then the universe temperature T rises in this period because of the cosmological invariant (28). As $\Pi = 0$ after the transition period the specific entropy was produced only in this period. Therefore, one can integrate the relation (54) for find the produced specific entropy.
It is given by

$$\sigma_\alpha - \sigma_{\alpha'} = \frac{4}{\ell}\left(\frac{1}{v_\alpha^2} - \frac{1}{v_{\alpha'}^2}\right) \tag{86}$$

where the quantities with their subscripts represent their values at the corresponding epoch times. Since $\sigma_{\alpha'} \ll \sigma_\alpha$ and $v_\alpha \ll v_{\alpha'}$, we have

$$\sigma_\alpha \simeq \frac{4}{\ell v_\alpha^2} \tag{87}$$

We can now specify the constant ℓ. The relation (87) would be exact if at the beginning epoch α' of the transition era, the specific entropy was $\sigma_{\alpha'} = 4/\ell v_{\alpha'}^2$, where $v_{\alpha'}^2$ satisfies the inequality condition (19). Here, this condition is given by

$$v_{\alpha'}^2 \leq 2 - \gamma = 1 - f(\theta) \simeq \frac{2}{3} \quad \left(\text{since } f(\theta) \simeq \frac{1}{3}\right) \tag{88}$$

Thus, one can take $v_{\alpha'}^2 \simeq 2/3$ and consequently we have

$$\sigma_{\alpha'} \simeq \frac{6}{\ell} \tag{89}$$

On the other hand, we have the constant specific entropy for the creation era (the mild inflation period upto the epoch α') as

$$\sigma = \frac{h - \mu n}{Tn} = \frac{h}{Tn} \quad (\text{assuming } \mu = 0 \text{ for this period})$$

or

$$\sigma = \frac{\gamma\rho}{p} = \frac{\gamma}{\gamma - 1} = \frac{1 + f(\theta)}{f(\theta)} \tag{90}$$

Since $f(\theta) \simeq 1$ for this period, we get $\sigma = 2$. This constant value of specific entropy changed at the beginning epoch α' of the transition era as $f(\theta)$ changed to a value 1/3. The value of σ at α' follows from (90). It is given by $\sigma_{\alpha'} = 4$. Therefore, from (89) it follows that $\ell = 3/2$ and consequently $\ell v_{\alpha'}^2 = 1$. Again, from (28) we have

$$v_\alpha^2 T_\alpha R_\alpha = v_{\alpha'}^2 T_{\alpha'} R_{\alpha'}$$

Therefore, from (87) it follows that

$$\sigma_\alpha = \frac{4}{\ell} \frac{T_\alpha}{T_{\alpha'}} \frac{R_\alpha}{R_{\alpha'} v_{\alpha'}^2} = 4 \frac{T_\alpha R_\alpha}{T_{\alpha'} R_{\alpha'}} \tag{91}$$

We have pointed out above that $T_{\alpha'} \ll T_\alpha$ where T_α is given in (80). The temperature $T_{\alpha'}$ can be computed from (76) if the beginning epoch α' of the transition period is ascertainable. However, one can find an estimate for the produced specific entropy by taking α' very close to α. If we take $\alpha' = 0.9\alpha$ we have $T_{\alpha'} \simeq 1.2 \times 10^{13}$ cm^{-1} and

$$\sigma_\alpha \simeq 2 \times 10^{10} \tag{92}$$

Even, a more short transition period (say, with $\alpha' = 0.99\alpha$) cannot change this estimate. On the other hand, a bit longer period with $\alpha' = 0.5\alpha$, say, cannot lower it very much. In fact, for this case we get $\sigma_\alpha \simeq 1.41 \times 10^{10}$. The estimate of σ_α corresponds to the present value of the specific entropy as it remained constant after the transition period. Also, it is to be noted that the temperature after the epoch α

did not increase further as v remained constant on the onset of the usual radiation-dominated FRW universe. In fact, it began to decrease because of the cosmological invariant (28) with constant v. The temperature at $t = \alpha$ can give the correct background temperature of the present universe. It is evident that the present value of the specific entropy depends on the duration and beginning of the transition period. Of course, the variation in its estimates is not much and as we have seen above that the estimates are of the order of 10^{10} in consistency with the observations.

Lastly, we calculate the total entropy in the observable universe with the assumption that nearly all entropy was produced just before the onset of standard radiation-dominated FRW era at the epoch α. At the epoch α, the total entropy in a comoving volume is

$$S = \sigma n_m(\alpha) R^3(\alpha) \tag{93}$$

To calculate S, we use the following conventional relation

$$\sigma = 3.7\,\frac{\rho_\gamma}{\rho_m}\frac{m}{T} = 3.7\frac{\rho_\gamma}{n_m T} \tag{94}$$

and the cosmological invariant (28) with constant v.
The cosmological invariant gives

$$\begin{aligned} R(\alpha)T(\alpha) &= R_0 T_0 = \text{constant} \\ &= 1.18 \times 10^{29} u (1 < u < 1.8) \end{aligned} \tag{95}$$

where, R_0 and T_0 are the present scale factor and the temperature of the universe, respectively. Again, ρ_γ and T at the epoch α are, respectively, given in (79) and (80). Now we have from (93) and (94),

$$S = 3.7\frac{\rho_\gamma}{T} R^3(\alpha) \tag{96}$$

From (95), using the value of T at $t = \alpha$, we have

$$R(\alpha) = 2.14 \times 10^6 u \,\text{cm} \tag{97}$$

Consequently, we have from (96),

$$S = 4.02 \times 10^{87} u^3 \tag{98a}$$

or, we have

$$4.02 \times 10^{87} < S < 2.34 \times 10^{88} \tag{98b}$$

From (93) and the value of σ_α as in (92) we can find the total particle number of the observable universe. In fact, it is nearly equal to the total particle number in a comoving volume at the epoch α. It is given by

$$N_m = n_m(\alpha)\; R^3(\alpha) = \frac{S}{\sigma_\alpha} = 2.01 \times 10^{77} u^3 \tag{99a}$$

that is,

$$2.01 \times 10^{77} < N_m < 1.17 \times 10^{78} \tag{99b}$$

These values for the entropy and particle number are in agreement with the accepted values (Kolb and Turner, 1990).

5. CONCLUDING REMARKS

In this article we have discussed full Israel–Stewart causal theory of bulk viscosity as applied to the imperfect fluid model of the universe. The universe regarded as a thermodynamically open system in the sense of Prigogine (1989) went through a mild inflationary stage due to the instability of the Minkowski space–time caused by the very massive particles (of masses about 50 times the Planck mass) at the Planck order time. Earlier (De, 1993b) it was shown that these massive particles were produced at this earliest epoch because of an anisotropic perturbation of the Minkowski space–time. The created particles could give the energy density of the universe at the Planck order time as well as in the subsequent era. Matter and entropy creation continued in the mild inflation era, which is a matter-dominated FRW stage. This stage transited into the standard radiation-dominated FRW era through an intermediate transition period when the particles (massive) became relativistic and contributed to the radiation energy density. This period was, of course, a phase transition modelled in terms of dissipative process and realized as viscous fluid model of the universe, where full causal theory of bulk viscosity has been applied.

We have shown the following results for the full causal theory as applied to the viscous fluid of transition era:

(i) The full (nontruncated) theory can be moulded into the form of the noncausal theory with an auxiliary condition which states that the square of dissipative contribution to the speed of sound, v^2, is proportional to the particle number in a comoving volume.

(ii) For the validity of the second law of thermodynamics the particle number (and also v) cannot increase if the relativistic chemical potential μ is nonzero. If $\mu = 0$, the second law of thermodynamics can be valid.

(iii) A cosmological invariant (28) has been shown to hold good. From this it follows that the temperature can increase in this era if the rate of decrease of v^2 is greater than that of increase of the scale factor $R(t)$. Also, the usual cosmological invariant follows with a generalized temperature. Again, the decrease of v ensures the validity of the second law of thermodynamics.

(iv) With this generalized temperature, the Gibbs equation has been generalized. It is also written in an alternative form in which an additional term has appeared. This additional term depends on the bulk viscosity like the cases of generalized Gibbs equation of other authors (Gariel and Le Denmat, 1994; Pavon *et al.*, 1982; Zakari and Jou, 1993). The rate of production of specific entropy has been obtained from this generalized Gibbs equation.

(v) For the transition era of the universe which is a thermodynamically open system of viscous fluid it is shown here that

$$\Pi + p_c \simeq 0$$

For positive p_c, we have a decrease in particle number and consequently we have $\Pi < 0$. This is the condition to be satisfied by the viscous stress because the dissipation due to it must give rise to a decrease in pressure. At the end of the transition era $p_c = 0$ as the particle number remained constant thereafter, and hence we have $\Pi = 0$ also.

It is mentioned here that our consideration of dissipative expansion satisfies the nonthermalizing condition. This is a consistency condition on causal viscous cosmologies, given in Maartens (1995). In our case, this condition is given by

$$\Gamma \sim \tau^{-1} = \frac{v^2 \gamma \rho}{\zeta(t)} < H \tag{100}$$

where Γ is the crucial interaction rate of the viscous fluid and is supposed to be determined by the characteristic time τ. It is easy to see that this condition is satisfied in our case.

We have found an estimate of the specific entropy as generated at the end of the transition era. This estimate corresponds to its present value in consistency with the observations. It is to be noted that the temperature increased from a value of nearly 10^{13} cm^{-1} to a large value of nearly 10^{22} cm^{-1} during the transition period. We have obtained the value of temperature at the epoch α not from the present value of it. On the other hand, it is calculated from the energy density at α, which is obtained from the energy density at the Planck order time. This energy density, in fact, is calculated from the quantum generation of massive particles due to the anisotropic perturbations of the Minkowski space–time at the very beginning epoch of the universe. We have also found the total entropy and the particle number of the observable universe in consistency with their accepted values. In this connection, we point out that Maartens (1995) has obtained such an accepted value of the total entropy via dissipative inflation without re-heating. Of course, no such estimates for the specific entropy and the total particle number are given therein.

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